arXiv:q-alg/9603005v1 8 Mar 1996

GENERALIZED WEIGHT FUNCTIONS AND THE MACDONALD POLYNOMIALS

T.H. Baker[1] and P.J. Forrester[2]
Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia

A weight function which q-generalizes the ground state wave function of the multi-component Calogero-Sutherland quantum many body system is introduced. Conjectures, and some proofs in special cases, are given for a constant term identity involving this function. A Gram-Schmidt procedure with respect to the inner product associated with the weight function is used to define orthogonal polynomials in one of the components, which are conjectured to be the Macdonald polynomials $P_\kappa(w_1,\dots,w_{N_0};qt^p,t)$, and a proof is given in a special case. Conjectures are also given for an adjoint property of the Macdonald operator with respect to the inner product associated with the weight function, and the normalization of the Macdonald polynomial with respect to the same inner product.

1. INTRODUCTION

In two recent studies [6, 7] of the multi-component Calogero-Sutherland model (quantum many body system with $1/r^2$ pair potential), one of us has been led to formulate a number of conjectures concerning Jack polynomials [14, 12] and the function

$$
\begin{aligned}
&|\psi_0(\{z_j^{(\alpha)}\}_{\substack{\alpha=1,\dots,p\\ j=1,\dots,N_\alpha}},\{w_j\}_{j=1,\dots,N_0})|^2\\
&:= \prod_{\alpha=1}^{p}\prod_{1\le j<k\le N_\alpha}|z_j^{(\alpha)}-z_k^{(\alpha)}|^{2\lambda+2}\prod_{1\le j'<k'\le N_0}|w_{k'}-w_{j'}|^{2\lambda}\\
&\quad\times\prod_{1\le\alpha<\beta\le p}\prod_{j=1}^{N_\alpha}\prod_{k=1}^{N_\beta}|z_j^{(\alpha)}-z_k^{(\beta)}|^{2\lambda}\prod_{\alpha=1}^{p}\prod_{j=1}^{N_\alpha}\prod_{j'=1}^{N_0}|z_j^{(\alpha)}-w_{j'}|^{2\lambda}, \qquad (1.1)
\end{aligned}
$$

where $w_j := e^{2\pi i y_j}$ and $z_j^{(\alpha)} := e^{2\pi i x_j^{(\alpha)}}$, which is the absolute value squared of the ground state wave function. In fact it appears that the Jack polynomials can be constructed via a Gram-Schmidt procedure based on (1.1) as a weight function.

Explicitly, define an inner product by

$$
\langle f|g\rangle_{N_0,\dots,N_p;\lambda} := \prod_{l=1}^{N_0}\int_{-1/2}^{1/2}dy_l\prod_{\alpha=1}^{p}\prod_{l=1}^{N_\alpha}\int_{-1/2}^{1/2}dx_l^{(\alpha)}\,|\psi_0(\{z_j^{(\alpha)}\}_{\substack{\alpha=1,\dots,p\\ j=1,\dots,N_\alpha}},\{w_j\}_{j=1,\dots,N_0})|^2 f^*\,g. \qquad (1.2)
$$

(this notation differs from that used in ref. [7] in that the weight function is explicitly included in the r.h.s.). Let κ denote a partition and define a symmetric polynomial in the variables $w_1,\dots,w_{N_0}$, denoted $p_\kappa(w_1,\dots,w_{N_0})$, by the following properties:

[1]email: tbaker@maths.mu.oz.au; supported by the ARC
[2]email: matpjf@maths.mu.oz.au; supported by the ARC

(i) $p_\kappa(w_1,\dots,w_{N_0}) = m_\kappa + \sum_{\mu<\kappa} a_\mu m_\mu$, where $|\mu| = |\kappa|$, $\mu < \kappa$ is with respect to reverse lexicographical ordering of the partitions, m_μ refers to the monomial symmetric function with exponents $\mu = (\mu_1,\dots,\mu_N)$ in the variables $w_1,\dots,w_{N_0}$ and a_μ is the corresponding coefficient;

(ii) for all $N_1,\dots,N_p \ge \kappa_1 - 1$, $\langle p_\kappa | p_\sigma \rangle_{N_0,\dots,N_p;\lambda} = 0$ for $\kappa \ne \sigma$.

(in ref. [7] condition (ii) required $N_1,\dots,N_p \ge \kappa_1$; this was weakened to the above statement in Conjecture 2.2 of ref. [7]). Then, according to Conjecture 2.4 of ref. [7], the polynomials p_κ are given in terms of the Jack polynomials by

$$p_\kappa(w_1,\dots,w_{N_0}) = J_\kappa^{(1+1/\lambda)}(w_1,\dots,w_{N_0}) \tag{1.3}$$

(here the normalization of J_κ is chosen so that the coefficient of m_κ is unity).

Conjecture 2.4 of ref. [7] is to be contrasted with the known theorem [14, 12] for the construction of the Jack polynomials via a Gram-Scmidt procedure based on the $p = 0$ cases of the weight function (1.1) and the inner product (1.2). Then the symmetric polynomials with properties (i) and (ii) above (in (ii), since $p = 0$, there is no restriction on κ_1) are given by

$$p_\kappa(w_1,\dots,w_{N_0}) = J_\kappa^{(1/\lambda)}(w_1,\dots,w_{N_0}). \tag{1.4}$$

Note that the parameter of the Jack polynomial here is $1/\lambda$.

The theory of Jack polynomials has been q-generalized by Macdonald [4] to give a theory of what are now referred to as Macdonald polynomials. This has motivated us to seek q-generalizations of the conjectures (and some theorems) contained in refs. [6, 7]. We begin in Section 2 by q-generalizing the weight function (1.1) and considering the q-generalizations of conjectured constant term identities given in ref. [6]. In Section 3 proofs of the conjectures of Section 2 are provided in certain cases. The Gram-Schmidt procedure is used with the q-generalization of (1.1) to define q-generalizations of the polynomials (1.3) in Section 4, and a conjecture is given relating these polynomials to the Macdonald polynomials. In fact we are led to conclude that (1.3) is only correct for $p = 1$. For general p we have new evidence which suggests that Conjecture 2.4 of ref. [7] should read

$$p_\kappa(w_1,\dots,w_{N_0}) = J_\kappa^{(p+1/\lambda)}(w_1,\dots,w_{N_0}). \tag{1.5}$$

We also provide a conjecture for a normalization integral involving the Macdonald polynomials, generalizing the conjecture given in ref. [7] in the Jack polynomial case. In Section 5 some proofs of the conjectures of Section 4 are provided in certain cases, while our results are briefly summarized in Section 6. In the Appendix we use a known generalization of the so called q-Morris theorem (see e.g. ref. [16]), which is the $p = 0$ case of the constant term identities considered in Section 2, to derive the expansion of the power sums in terms of Macdonald polynomials.

2. THE q-GENERALIZED WEIGHT FUNCTION AND CONSTANT TERM IDENTITIES

2.1 Revision of the case $p = 0$

For $p = 0$ and λ integer, since $|w_j| = 1$, (1.1) can be written as

$$|\psi_0(\{w_j\}_{j=1,\dots,N_0})|^2 = \prod_{1\le j<k\le N_0} \Big(1 - \frac{w_k}{w_j}\Big)^\lambda \Big(1 - \frac{w_j}{w_k}\Big)^\lambda. \tag{2.1}$$

This was first q-generalized by Andrews [1] as

$$|\psi_0(\{w_j\}_{j=1,\dots,N_0}; q)|^2 := \prod_{1\le j<k\le N_0} \Big(q\frac{w_k}{w_j}; q\Big)_\lambda \Big(\frac{w_j}{w_k}; q\Big)_\lambda \tag{2.2}$$

where

$$(a;q)_\lambda := \prod_{l=0}^{\lambda-1}(1-aq^l), \quad \lambda \in Z_{\geq 0}. \tag{2.3}$$

The criterium used to choose this q-generalization (note that unlike (2.1), (2.2) is not symmetric in $w_1,\dots,w_{N_0}$), additional to requiring that (2.2) reduces to (2.1) when $q=1$, was that the Dyson identity [5]

$$\mathrm{CT}\,|\psi_0(\{w_j\}_{j=1,\dots,N_0})|^2 = \frac{(\lambda N_0)!}{\lambda!_0^N}, \quad \lambda \in Z_{\geq 0}, \tag{2.4}$$

generalizes as

$$\mathrm{CT}\,|\psi_0(\{w_j\}_{j=1,\dots,N_0};q)|^2 = \frac{\Gamma_q(\lambda N_0+1)}{(\Gamma_q(\lambda+1))_0^N}, \quad \lambda \in Z_{\geq 0} \tag{2.5}$$

where

$$\Gamma_q(n+1) := \prod_{j=1}^{n}\frac{1-q^j}{1-q} \tag{2.6}$$

and CT denotes the constant term in the Laurent polynomial. Note the restriction $\lambda \in Z_{\geq 0}$ in the above formulas. For general λ we interpret (2.3) as

$$(a;q)_\lambda := \frac{(a;q)_\infty}{(aq^\lambda;q)_\infty}. \tag{2.7}$$

As pointed out by Stembridge [15], the identity (2.5) still holds with

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} = \frac{(q;q)_{x-1}}{(1-q)^{x-1}} \tag{2.8}$$

and

$$\mathrm{CT}\,f(w_1,\dots,w_N) = \prod_{l=1}^{N}\int_{-1/2}^{1/2}dx_l\, f(e^{2\pi i x_1},\dots,e^{2\pi i x_N}). \tag{2.9}$$

2.2 q-generalization for general p

Motivated by the q-generalization (2.2) of (2.1), we formulated the q-generalization of (1.1) as

$$\begin{aligned}
&|\psi_0(\{z_j^{(\alpha)}\}_{\substack{\alpha=1,\dots,p\\ j=1,\dots,N_\alpha}},\{w_j\}_{j=1,\dots,N_0};q)|^2 \\
&= \prod_{\alpha=1}^{p}\prod_{1\leq j<k\leq N_\alpha}\Big(\frac{z_j^{(\alpha)}}{z_k^{(\alpha)}};q\Big)_{\lambda+1}\Big(q\frac{z_k^{(\alpha)}}{z_j^{(\alpha)}};q\Big)_{\lambda+1}\prod_{1\leq j'<k'\leq N_0}\Big(\frac{w_{j'}}{w_{k'}};q\Big)_\lambda\Big(q\frac{w_{k'}}{w_{j'}};q\Big)_\lambda \\
&\quad\times\prod_{1\leq\alpha<\beta\leq p}\prod_{j=1}^{N_\alpha}\prod_{k=1}^{N_\beta}\Big(\frac{z_j^{(\alpha)}}{z_k^{(\beta)}};q\Big)_\lambda\Big(q\frac{z_k^{(\beta)}}{z_j^{(\alpha)}};q\Big)_\lambda\prod_{\alpha=1}^{p}\prod_{j=1}^{N_\alpha}\prod_{j'=1}^{N_0}\Big(\frac{z_j^{(\alpha)}}{w_{j'}};q\Big)_\lambda\Big(q\frac{w_{j'}}{z_j^{(\alpha)}};q\Big)_\lambda.
\end{aligned} \tag{2.10}$$

Indeed this q-generalization appears to generalize an integration formula for (1.1), conjectured in ref. [6], in the same way that (2.5) generalizes (2.4).

To be more explicit, let us consider the case $p=1$, and introduce the notation

$$\begin{aligned}
&D_p(N_1;N_0;a,b,\lambda) \\
&:= \left(\prod_{l=1}^{N_0}\int_{-1/2}^{1/2}dx_l\, w_l^{(a-b)/2}|1+w_l|^{a+b}\right)\left(\prod_{l=1}^{N_1}\int_{-1/2}^{1/2}dx_l\, z_l^{(a-b)/2}|1+z_l|^{a+b}\right) \\
&\quad\times|\psi_0(\{z_j\}_{j=1,\dots,N_1},\{w_j\}_{j=1,\dots,N_0})|^2.
\end{aligned} \tag{2.11}$$

In ref. [6, eq.(3.21)] it was conjectured that

$$
\begin{aligned}
&D_1(N_1;N_0;a,b,\lambda)\\
&\quad= \prod_{j=0}^{N_1-1} \frac{(j+1)\Gamma((\lambda+1)j+a+b+\lambda N_0+1)\Gamma((\lambda+1)(j+1)+\lambda N_0)}{\Gamma(1+\lambda)\Gamma((\lambda+1)j+a+\lambda N_0+1)\Gamma((\lambda+1)j+b+\lambda N_0+1)}\\
&\qquad\times \prod_{l=0}^{N_0-1} \frac{\Gamma(a+b+1+\lambda l)\Gamma(1+\lambda(l+1))}{\Gamma(a+1+\lambda l)\Gamma(b+1+\lambda l)\Gamma(1+\lambda)}
\end{aligned}
\tag{2.12}
$$

To formulate the q-generalization of (2.11) we note that for a and b integers and $|u|=1$

$$
u^{(a-b)/2}|1+u|^{a+b} = (1+u)^a(1+1/u)^b. \tag{2.13}
$$

This suggests we define

$$
\begin{aligned}
&D_1(N_1;N_0;a,b,\lambda;q)\\
&\quad:= \prod_{l=1}^{N_0}\int_{-1/2}^{1/2} dy_l\,(-w_l;q)_a\Big(-\frac{q}{w_l};q\Big)_a \prod_{l=1}^{N_1}\int_{-1/2}^{1/2} dx_l\,(-z_l;q)_b\Big(-\frac{q}{z_l};q\Big)_b\\
&\qquad\times|\psi_0(\{z_j\}_{j=1,\dots,N_1},\{w_j\}_{j=1,\dots,N_0};q)|^2\\
&\quad= \prod_{l=1}^{N_0}\int_0^1 dy_l\,(w_l;q)_a\Big(\frac{q}{w_l};q\Big)_a \prod_{l=1}^{N_1}\int_0^1 dx_l\,(z_l;q)_b\Big(\frac{q}{z_l};q\Big)_b\\
&\qquad\times \prod_{1\le j<k\le N_1}\Big(\frac{z_j}{z_k};q\Big)_{\lambda+1}\Big(q\frac{z_k}{z_j};q\Big)_{\lambda+1} \prod_{1\le j'<k'\le N_0}\Big(\frac{w_{j'}}{w_{k'}};q\Big)_\lambda\Big(q\frac{w_{k'}}{w_{j'}};q\Big)_\lambda\\
&\qquad\times \prod_{j=1}^{N_1}\prod_{j'=1}^{N_0}\Big(\frac{z_j}{w_{j'}};q\Big)_\lambda\Big(q\frac{w_{j'}}{z_j};q\Big)_\lambda
\end{aligned}
\tag{2.14}
$$

On the basis of exact computer generated data (obtained for $\lambda=1$ and 2, with various ‘small’ values of N_0, N_1, a and b) and some analytic evaluations in certain special cases (presented in the next section) we make the following conjecture for the evaluation of (2.11).

Conjecture 2.1 We have

$$
\begin{aligned}
&D_1(N_1;N_0;a,b,\lambda)\\
&\quad= \frac{\Gamma_{q^{\lambda+1}}(N_1+1)}{(\Gamma_q(1+\lambda))^{N_0+N_1}} \prod_{j=0}^{N_1-1} \frac{\Gamma_q((\lambda+1)j+a+b+\lambda N_0+1)\Gamma_q((\lambda+1)(j+1)+\lambda N_0)}{\Gamma_q((\lambda+1)j+a+\lambda N_0+1)\Gamma_q((\lambda+1)j+b+\lambda N_0+1)}\\
&\qquad\times \prod_{l=0}^{N_0-1} \frac{\Gamma_q(a+b+1+\lambda l)\Gamma_q(1+\lambda(l+1))}{\Gamma_q(a+1+\lambda l)\Gamma_q(b+1+\lambda l)}
\end{aligned}
\tag{2.15}
$$

Note that the ‘base’ of the q-gamma function in the denominator of the first term is $q^{\lambda+1}$ whereas in all other terms it is q. Also, when $N_1=0$, note that this reduces to the so-called q-Morris theorem (see e.g. ref. [16]).

2.3 The q-generalized integral for general p

In the $q=1$ case it was conjectured [6, eq.(4.8a)] that for general p the analogue of the integral (2.14), D_p say, satisfies a functional equation. Using this functional equation it was shown D_p can be uniquely determined by a recurrence. To q-generalize this result, let

$$
D_p(N_1,\dots,N_p;N_0;a,b,\lambda;q)
$$

$$:= \prod_{l=1}^{N_0} \int_{-1/2}^{1/2} dy_l \prod_{\alpha=1}^{p} \prod_{l=1}^{N_\alpha} \int_{-1/2}^{1/2} dx_l^{(\alpha)} \, A(\{-z_j^{(\alpha)}\}_{\substack{\alpha=1,\dots,p \\ j=1,\dots,N_\alpha}}, \{-w_j\}_{j=1,\dots,N_0}; q)$$
$$\times |\psi_0(\{z_j^{(\alpha)}\}_{\substack{\alpha=1,\dots,p \\ j=1,\dots,N_\alpha}}, \{w_j\}_{j=1,\dots,N_0}; q)|^2 \tag{2.16a}$$

where

$$A(\{z_j^{(\alpha)}\}_{\substack{\alpha=1,\dots,p \\ j=1,\dots,N_\alpha}}, \{w_j\}_{j=1,\dots,N_0}; q) := \prod_{l=1}^{N_0} (w_l;q)_a \Big(\frac{q}{w_l};q\Big)_a \prod_{\alpha=1}^{p} \prod_{j=1}^{N_\alpha} (z_l^{(\alpha)};q)_b \Big(\frac{q}{z_l^{(\alpha)}};q\Big)_b. \tag{2.16b}$$

Guided by the conjecture in ref. [6, eq.(4.8a)], and Conjecture 2.1 above, we can make the following conjecture for the general p case.

Conjecture 2.2 For $N_p \geq N_j - 1$ $(j = 1, \dots, p-1)$ we have

$$\frac{D_p(N_1, \dots, N_{p-1}, N_p + 1; N_0; a, b, \lambda; q)}{D_p(N_1, \dots, N_{p-1}, N_p; N_0; a, b, \lambda; q)} = \frac{[N_p + 1]_{q^{\lambda+1}}}{\Gamma_q(\lambda + 1)}$$
$$\times \frac{\Gamma_q((\lambda+1)N_p + a + b + \lambda\sum_{j=0}^{p-1} N_j + 1)\Gamma_q((\lambda+1)(N_p+1) + \lambda\sum_{j=0}^{p-1} N_j)}{\Gamma_q((\lambda+1)N_p + a + \lambda\sum_{j=0}^{p-1} N_j + 1)\Gamma_q((\lambda+1)N_p + b + \lambda\sum_{j=0}^{p-1} N_j + 1)}. \tag{2.17}$$

In Conjecture 2.2 we have introduced the notation

$$[a]_q := \frac{1-q^a}{1-q}. \tag{2.18}$$

In the limit $q \to 1$ the formula in Conjecture 2.2 is equivalent to the functional equation conjectured in ref. [6, eq.(4.8a)].

3. ANALYTIC EVALUATIONS OF THE q-GENERALIZED INTEGRAL

In this section Conjecture 2.1 will be proved in some special cases.

3.1 The case $p = 1$, $a = \lambda$ and general N_0, N_1, b, λ

It was noted in ref.[6] that the integral evaluation in Conjecture 2.1 for $q = 1$ in the case $p = 1$, $a = b = 0$ and general N_0, N_1, λ follows from a theorem of Bressoud and Goulden [3]. This theorem has a q-counterpart, obtained by the same authors in an earlier publication.

Proposition 3.1 [3, Proposition 2.4, with A replaced by $\bar{A}$, the complement of A, to be consistent with the formulation in ref. [2]] Let $a_1, \dots, a_N$ be positive integers, A be an arbitrary subset of $\{(i,j) : 1 \leq i < j \leq n\}$, G_A be the set of permutations σ on $\{1, \dots, n\}$ (with $\sigma(i) := \sigma_i$) whose inversions $I(\sigma) := \{(\sigma_i, \sigma_j) : j > i \quad \text{and} \quad \sigma_j < \sigma_i\}$ are contained in A:

$$G_A = \{\sigma : \text{ if } (j,i) \in I(\sigma), \text{ then } (i,j) \in A\},$$

and let $\chi(T)$ be the characteristic function which is 1 if T is true, 0 otherwise. We have

$$\text{CT} \prod_{1 \leq i < j \leq n} \Big(q\frac{x_i}{x_j};q\Big)_{a_i} \Big(\frac{x_j}{x_i};q\Big)_{a_j - \chi((i,j)\notin A)} = \frac{\Gamma_q(a_1 + \ldots + a_n + 1)}{\Gamma_q(a_1)\ldots\Gamma_q(a_n)} S_n(\{a_j\}_{j=1,\dots,n}; G_A),$$

where

$$S_n(\{a_j\}_{j=1,\dots,n}; G_A) := \sum_{\sigma \in G_A} q^{\sum_{(i,j)\in I(\sigma)} a_i} \prod_{l=1}^{n} \frac{1-q}{1-q^{a_{\sigma_1} + \ldots + a_{\sigma_l}}}.$$

In this theorem, suppose $n = 1 + N_0 + N_1$,

$$A = \{(i,j) : 1 \le i < j \le N_0 + 1 \quad \text{or} \quad N_1 + 2 \le i < j \le N_0 + N_1 + 1\},$$

$$a_1 = b, \quad , a_2 = \ldots = a_{N_0+1} = \lambda, \quad a_{N_0+2} = \ldots a_{N_0+N_1+1} = \lambda + 1, \tag{3.1}$$

and replace $x_2, \ldots, x_{N_0+N_1+1}$ by $x_1x_2, \ldots, x_1x_{N_0+N_1+1}$. We see that the l.h.s. is of the form (2.14) with $a = \lambda$ and b, λ arbitrary positive integers. Proposition 3.1 therefore gives

$$D_1(N_1; N_0; \lambda, b, \lambda; q) = \frac{\Gamma_q((\lambda+1)N_1 + \lambda N_0 + b + 1)}{\Gamma_q(b)(\Gamma_q(\lambda))^{N_0}(\Gamma_q(\lambda+1))^{N_1}} S_{N_0,N_1}(\{a_j\}_{j=1,\ldots,n}; G_A). \tag{3.2}$$

Our proof of Conjecture 2.1 for the evaluation of $D_1(N_1; N_0; \lambda, b, \lambda; q)$ now follows from the following evaluation of S_{N_0,N_1}.

Proposition 3.2 With A, $\{a_j\}$ given by (3.1) we have

$$S_{N_0,N_1}(\{a_j\}; G_A) = \frac{1}{[b]_q[\lambda]_q^{N_0}[\lambda+1]_q^{N_1}} \prod_{j=1}^{N_1} \frac{[(\lambda+1)j]_q}{[(\lambda+1)j + \lambda N_0 + b]_q}.$$

Proof We will establish a recurrence relation in N_1. To solve the recurrence we require the value of $S_{N_0,0}$. This is obtained by noting that when $N_1 = 0$, D_1 is given by the q-Morris theorem (Conjecture 2.1 with $N_1 = 0$). Comparison with (3.2) then gives

$$S_{N_0,0}(\{a_j\}; G_A) = \frac{1}{[b]_q[\lambda]_q^{N_0}}. \tag{3.3}$$

The recurrence is obtained by noting that all permutations in G_A are of the form $\sigma = (\sigma', \sigma'')$, where σ' is a permutation of $\{1, \ldots, N_0+1\}$, and σ'' is a permutation of $\{N_0+2, \ldots, N_0+N_1+1\}$. Thus, if N_1 is increased by 1 only σ'' can be affected. Furthemore, for each $\sigma = (\sigma', \sigma'')$ in G_A before increasing N_1 by 1, there are N_1+1 permutations in G_A after N_1 is increased by 1, which are given by $\sigma = (\sigma', \sigma''|_{k \mapsto N_0+N_1+2}, k)$, $k = N_0+2, \ldots, N_0+N_1+2$ (for $k = N_0+N_1+2$, σ'' remains unchanged). Denote these permutations by $G_A(k)$ so that for N_1 increased by 1, $G_A = \cup_{k=N_0+2}^{N_1+N_0+2} G_A(k)$. The facts that the replacement $k \mapsto N_0+N_1+2$ in σ'' creates N_0+N_1+2-k new inversions, that $a_i = \lambda+1$ for $i = N_0+2, \ldots, N_0+N_1+2$, and that

$$(1-q)/(1-q^{a_{\sigma_1}+\ldots+a_{\sigma_{N_0+N_1+2}}}) = (1-q)/(1-q^{a_1+\ldots+a_{N_0+N_1+2}})$$

is a common factor in the summand now gives the recurrence

$$S_{N_0,N_1+1}(\{a_j\}; G_A(k)) = \frac{q^{(\lambda+1)(N_0+N_1+2-k)}}{[(\lambda+1)(N_1+1) + \lambda N_0 + b]_q} S_{N_0,N_1}(\{a_j\}; G_A(k))$$

Summing over k we have

$$S_{N_0,N_1+1}(\{a_j\}; G_A) = \frac{[(\lambda+1)(N_1+1)]_q}{[(\lambda+1)(N_1+1) + \lambda N_0 + b]_q[\lambda+1]_q} S_{N_0,N_1}(\{a_j\}; G_A),$$

which upon iteration and use of the initial condition gives the stated result.

Substituting the result of Proposition 3.2 in (3.2) evaluates $D_1(N_1; N_0; \lambda, b, \lambda; q)$ for λ and b arbitrary positive integers; by a simple lemma of Stembridge [15, lemma 3.2], the validity of the positive integer case implies the validity for all (complex) λ and b. Comparison between the

resulting expression for D_1 and the expression of Conjecture 3.1 in the case $a=\lambda$ shows, after simplification of the latter, that the two expressions are identical.

3.2 The case $p=1$, $N_1=2$ and general N_0, N_1, a, b, λ

We address the case $N_1=2$, as this is the first non-trivial case; when $N_1=1$, it is clear that $D_1(1;N_0;a,b,\lambda;q)=D_0(N_0+1;a,b,\lambda;q)$. To prove Conjecture 2.1 in this particular case, we adopt the method of Stembridge and Stanton[15]. The essence of this method, applied to the problem at hand, is to express the constant term of the two-component function in terms of the constant term of the one-component function (that is, the function appearing in the q-Morris identity), by means of a partial expansion of the two-component function.

It will prove useful to briefly summarize the results of Zeilberger [16] as we shall be aiming to extend his proof given there, of the q-Morris identity. Let $[x^0]f(x_1,\dots,x_n) := \mathrm{CT} f(x_1,\dots,x_n)$, and in general let $[x^\beta]f(x_1,\dots,x_n)$ denote the coefficient of $x^\beta := x_1^{\beta_1}x_2^{\beta_2}\cdots x_n^{\beta_n}$ in the expansion of f. For notational convenience, this can be extended so that for a general function $g(x)=\sum_\beta a_\beta x^\beta$, one writes $[g]f := \sum_\beta a_\beta [x^\beta]f$.

The "reduced" q-Morris identity takes the form

$$\begin{aligned}
[x^0]F_0(x) &= \prod_{j=0}^{n-1} \frac{\Gamma_q(\lambda j+a+b+1)\Gamma_q(\lambda(j+1))}{\Gamma_q(\lambda j+a+1)\Gamma_q(\lambda j+b+1)\Gamma_q(\lambda)} \\
&= \frac{1}{\Gamma_{q^\lambda}(n+1)}\frac{1}{(\Gamma_q(\lambda+1))^n}\prod_{j=0}^{n-1}\frac{\Gamma_q(\lambda j+a+b+1)\Gamma_q(\lambda(j+1)+1)}{\Gamma_q(\lambda j+a+1)\Gamma_q(\lambda j+b+1)}
\end{aligned} \tag{3.4}$$

where

$$F_0(x) := \prod_{i=1}^n (x_i;q)_a\,(\frac{q}{x_i};q)_b \prod_{1\le i<j\le n}(\frac{x_j}{x_i};q)_\lambda\,(q\frac{x_i}{x_j};q)_{\lambda-1} \tag{3.5}$$

A lemma of Stembridge [15] gives that (3.4) is equivalent to the original q-Morris identity i.e. (2.15) with $N_1=0$, $N_0=n$. Zeilberger's proof of the "reduced" q-Morris identity relies on the function $F_0(x)$ being almost anti-symmetric. Indeed, $F_0(x)=x^{-\delta}G_0(x)$ where $\delta := (n-1,n-2,\dots,2,1,0)$, and $G_0(x)$ *is* anti-symmetric. Thus, the constant term of the function $F_0(x)$ is just $[x^\delta]G_0(x)$. As part of his inductive proof, he essentially uses the equation

$$q^{\beta_n}\left[x^{\beta+\delta}(1-qux_n)\prod_{i=1}^{n-1}(1-t\frac{x_n}{x_i})\right]G_0(x) = t^{n-1}\left[x^{\beta+\delta}(s-x_n)\prod_{i=1}^{n-1}(1-t^{-1}\frac{x_n}{x_i})\right]G_0(x), \tag{3.6}$$

where $u:=q^b$, $s:=q^a$, $t:=q^\lambda$, to relate $[x^{\beta+\delta}]G_0(x)$ to $[x^\delta]G_0(x)$ for various special values of β. This is done by using the anti-symmetry of $G_0(x)$ and his "Crucial lemma"

Lemma 3.3[16] If $G_0(x_1,\dots,x_n)$ is an anti-symmetric Laurent polynomial, $\gamma\in\mathbb{Z}^n$, and σ a permutation then $[x^{\sigma(\gamma)}]G_0(x)=\operatorname{sgn}\sigma\,[x^\gamma]G_0(x)$. In particular, if any two components of γ are equal, then $[x^\gamma]G_0(x)=0$.

As an example of how this is done, let us give a result we shall use subsequently.

Lemma 3.4 Let $\alpha_1=(1,0,\dots,0)$ and $\alpha_2=(1,0,\dots,0,-1)$. Then

$$[x^{\alpha_1+\delta}]G_0 = \frac{(s-1)(1-t^n)}{(1-qut^{n-1})(1-t)}\,[x^\delta]G_0 \tag{3.7}$$

$$[x^{\alpha_2+\delta}]G_0 \quad = \quad \frac{(t-qs)(1-t^{n-1})}{(1-qst^{n-1})(1-t)}\,[x^{\delta}]G_0 + \frac{q(u-t)(1-t^{n-1})}{(1-qst^{n-1})(1-t)}\,[x^{\alpha_1+\delta}]G_0 \tag{3.8}$$

Proof Bearing in mind (3.6), first look at the expansion

$$\prod_{i=1}^{n-1}(1-z\frac{x_n}{x_i}) = \sum_T (-zx_n)^{|T|}x^{-T} \tag{3.9}$$

where the sum is over all $T \subseteq \{1,2,\ldots,n-1\}$, and $x^{-T} := \prod_{i\in T} x_i^{-1}$. For each m with $0 \le m \le n-1$, there exists a unique set T, such that $|T| = m$, and $x^{\delta}x_n^{|T|}x^{-T}$ has distinct exponents; namely $T = \{n-m, n-m+1, \ldots, n-1\}$. In fact $x^{\delta}x_n^{|T|}x^{-T} = x^{\sigma(\delta)}$ where σ is a permutation with $\operatorname{sgn}\sigma = (-1)^m$. Thus

$$\left[x^{\delta}\prod_{i=1}^{n-1}\left(1-z\frac{x_n}{x_i}\right)\right]G_0 = \sum_{m=0}^{n-1}(-z)^m(-1)^m\,[x^{\delta}]G_0 = \left(\frac{1-z^n}{1-z}\right)[x^{\delta}]G_0 \tag{3.10}$$

We must also expand

$$x_n\prod_{i=1}^{n-1}(1-z\frac{x_n}{x_i}) = \sum_T(-z)^{|T|}(x_n)^{|T|+1}x^{-T} \tag{3.11}$$

In this case, there is only one set T such that $x^{\delta}x_n^{|T|}x^{-T}$ has distinct exponents: $T = \{1,2,\ldots,n-1\}$. Moreover, for this set T,

$$x^{\delta}x_n^{|T|}x^{-T} = x_n^n x_1^{n-2}x_2^{n-3}\cdots x_{n-2}^1 x_{n-1}^0 = x^{\sigma(\alpha_1+\delta)}$$

where $\operatorname{sgn}\sigma = (-1)^{n-1}$. Thus

$$\left[x^{\delta}x_n\prod_{i=1}^{n-1}\left(1-z\frac{x_n}{x_i}\right)\right]G_0 = z^{n-1}[x^{\alpha_1+\delta}]G_0 \tag{3.12}$$

If one now uses (3.10) and (3.12) with $z = t,\, t^{-1}$ in (3.6), and sets $\beta = 0$, the stated result (3.7) follows.

To prove (3.8), note that the sets T such that $x^{\alpha_2+\delta}x_n^{|T|}x^{-T}$ has distinct exponents are of the form

$$T = \begin{cases} \emptyset & \\ \{1, n-m+1, n-m+2, \ldots, n-1\} & 1 \le m \le n-1 \end{cases}$$

in which case

$$x^{\alpha_2+\delta}x_n^n x^{-T} = \begin{cases} x^{\alpha_2+\delta} & \\ x^{\sigma(\delta)} & \operatorname{sgn}\sigma = (-1)^{m-1} \end{cases}$$

Thus

$$\left[x^{\alpha_2+\delta}\prod_{i=1}^{n-1}\left(1-z\frac{x_n}{x_i}\right)\right]G_0 = [x^{\alpha_2+\delta}]G_0 - z\left(\frac{1-z^{n-1}}{1-z}\right)[x^{\delta}]G_0 \tag{3.13}$$

Similarly, the sets T such that $x^{\alpha_2+\delta}x_n^{|T|+1}x^{-T}$ has distinct exponents are of the form $T = \{n-m, n-m+1, \ldots, n-1\}$, $0 \le m \le n-2$, in which case $x^{\alpha_2+\delta}x_n^{|T|+1}x^{-T} = x^{\sigma(\alpha_1+\delta)}$, with $\operatorname{sgn}\sigma = (-1)^m$. Hence

$$\left[x^{\alpha_2+\delta}x_n\prod_{i=1}^{n-1}\left(1-z\frac{x_n}{x_i}\right)\right]G_0 = \left(\frac{1-z^{n-1}}{1-z}\right)[x^{\alpha_1+\delta}]G_0 \tag{3.14}$$

Again, using (3.13), (3.14) in (3.6) (setting $\beta = \alpha_2$), with $z = t,\, t^{-1}$ yields (3.8).

Returning to the proof of the $N_1 = 2$ case of Conjecture 2.1, we first make the substitutions $w_i \to w_{N_0+1-i}$, $z_i \to z_{N_1+1-i}$ (which has no effect on the constant term) and then follow the arguments in ref [15], whereby we replace $(qz_i/z_j;q)_{\lambda+1} \to (qz_i/z_j;q)_\lambda$ and $(qw_i/w_j;q)_\lambda \to (qw_i/w_j;q)_{\lambda-1}$, to obtain an alternative statement of Conjecture 2.1 in the case $p = 2$ which reads as follows:

$$\begin{aligned}
&\mathrm{CT} \prod_{i=1}^{N_0} (w_i;q)_a (q/w_i;q)_b \prod_{j=1}^{N_1} (z_j;q)_a (q/z_j;q)_b \prod_{1\le i<j\le N_0} (\frac{w_j}{w_i};q)_\lambda (q\frac{w_i}{w_j};q)_{\lambda-1} \\
&\qquad\qquad \times \prod_{1\le i<j\le N_1} (\frac{z_j}{z_i};q)_{\lambda+1} (q\frac{z_i}{z_j};q)_\lambda \prod_{i=1}^{N_0}\prod_{j=1}^{N_1} (q\frac{w_i}{z_j};q)_\lambda (\frac{z_j}{w_i};q)_\lambda \\
&= \frac{1}{\Gamma_{q^\lambda}(N_0+1)\Gamma_{q^{\lambda+1}}(N_1+1)} D_1(N_1;N_0;a,b,\lambda)
\end{aligned} \tag{3.15}$$

where $D_1(N_1;N_0;a,b,\lambda)$ is given in (2.15). In the particular case of $N_1 = 2$, the function appearing on the left-hand side of the above equation, call it $F_1(\{z_j\};\{w_j\})$ say, is simply related to the $n = N_0 + 2$ variable function $F_0(x)$ in (3.5). Thus, letting $x_i = w_i$, $1 \le i \le N_0$, and $x_{N_0+1} = z_1$, $x_{N_0+2} = z_2$, we have

$$F_1(x) := (1 - t\frac{x_{N_0+1}}{x_{N_0+2}})(1 - t\frac{x_{N_0+2}}{x_{N_0+1}}) \prod_{i=1}^{N_0} (1 - t\frac{x_i}{x_{N_0+1}})(1 - t\frac{x_i}{x_{N_0+2}})\, F_0(x)$$

Using the "reduced" q-Morris identity (3.4), it suffices to prove

$$[x^0]F_1(x) = \frac{(1-t^{N_0+1})(1-t^{N_0+2})(1-qust^{N_0+1})(1-qt^{N_0+2})}{(1-t)^2(1-qut^{N_0+1})(1-qst^{N_0+1})}\, [x^0]F_0(x) \tag{3.16}$$

Note that we can rewrite $[x^0]F_1(x)$ in the following form,

$$[x^0]F_1(x) = \left[\left((1+t^2) - t\frac{x_{N_0+1}}{x_{N_0+2}} - t\frac{x_{N_0+2}}{x_{N_0+1}} \right) \prod_{i=1}^{N_0} \left(1 - t\frac{x_{N_0+1}}{x_i}\right)\left(1 - t\frac{x_{N_0+2}}{x_i}\right) x^\delta \right] G_0(x) \tag{3.17}$$

where $\delta = (N_0+1, N_0, \dots, 1, 0)$, and $G_0(x)$ is anti-symmetric. Let us now show that each of the terms $[A(x)\prod_{i=1}^{N_0}(1 - tx_{N_0+1}/x_i)(1 - tx_{N_0+2}/x_i)x^\delta]G_0$, for $A(x) = 1 + t^2$, $-tx_{N_0+1}/x_{N_0+2}$ and $-tx_{N_0+2}/x_{N_0+1}$ can be expressed in terms of $[x^\delta]G_0$ and $[x^{\alpha_2+\delta}]G_0$ using the above techniques.

Lemma 3.5 We have

$$\begin{aligned}
\left[\prod_{i=1}^{N_0} \left(1 - t\frac{x_{N_0+1}}{x_i}\right)\left(1 - t\frac{x_{N_0+2}}{x_i}\right) x^\delta \right] G_0 &= B_{N_0}(t) \\
\left[-t\frac{x_{N_0+2}}{x_{N_0+1}} \prod_{i=1}^{N_0} \left(1 - t\frac{x_{N_0+1}}{x_i}\right)\left(1 - t\frac{x_{N_0+2}}{x_i}\right) x^\delta \right] G_0 &= tB_{N_0}(t)
\end{aligned}$$

where

$$B_{N_0}(t) = \frac{1}{(1-t)} \left(\frac{(1-t^{N_0+1})}{(1-t)} - t\frac{(1-t^{2N_0+2})}{(1-t^2)} \right) [x^\delta]G_0$$

Proof We prove only the first formula, as the proof of the second is similar. First, expand

$$\prod_{i=1}^{N_0} \left(1 - t\frac{x_{N_0+1}}{x_i}\right)\left(1 - t\frac{x_{N_0+2}}{x_i}\right) = \sum_{n,m,\gamma} (-t)^{n+m} c_{n,m,\gamma}\, f_{n,m,\gamma}(x^{-1})\, x_{N_0+1}^n x_{N_0+2}^m$$

where $f_{n,m,\gamma}$ is the monomial $x_1^{\gamma_1}\cdots x_{N_0}^{\gamma_{N_0}}$ with exponents $\gamma_i = 0, -1$ or -2, and $c_{n,m,\gamma}$ is a positive integer. The only terms in this expansion which have distinct exponents when multiplied by x^δ occur when $n \geq m$. Moreover

$$f_{n,m,\gamma} = x_{N_0+1-n}^{-1} x_{N_0+2-n}^{-1} \cdots x_{N_0-m}^{-1} x_{N_0+1-m}^{-2} \cdots x_{N_0}^{-2},$$

$c_{n,m,\gamma} = 1$ and

$$f_{n,m,\gamma}(x^{-1}) x_{N_0+1}^{n} x_{N_0+2}^{m} x^\delta = x_1^{N_0+1} \cdots x_{N_0-n}^{n+2} x_{N_0+1}^{n+1} x_{N_0+1-n}^{n} \cdots x_{N_0-m}^{m+1} x_{N_0+2}^{m} x_{N_0+1-m}^{m-1} \cdots x_{N_0}^{0} = x^{\sigma(\delta)}$$

where $\operatorname{sgn}\sigma = (-1)^{n+m}$. Thus,

$$\left[\prod_{i=1}^{N_0}\left(1 - t\frac{x_{N_0+1}}{x_i}\right)\left(1 - t\frac{x_{N_0+2}}{x_i}\right) x^\delta\right] G_0 = \sum_{n=0}^{N_0}\sum_{m=0}^{n} (-1)^{n+m}(-t)^{n+m}[x^\delta] G_0$$

which yields the result upon carrying out the summation.

Lemma 3.6 We have

$$\left[-t\frac{x_{N_0+1}}{x_{N_0+2}}\prod_{i=1}^{N_0}\left(1 - t\frac{x_{N_0+1}}{x_i}\right)\left(1 - t\frac{x_{N_0+2}}{x_i}\right) x^\delta\right] G_0(x) = -t^{N_0+1}[x^{\alpha_2+\delta}]\, G_0(x)$$
$$-\frac{t^2}{1-t}\left(\frac{1-t^{2N_0}}{1-t^2} - t^{N_0}\frac{1-t^{N_0}}{1-t}\right)[x^\delta]\, G_0(x) \qquad (3.18)$$

Proof Again, expand

$$-t\frac{x_{N_0+1}}{x_{N_0+2}}\prod_{i=1}^{N_0}\left(1 - t\frac{x_{N_0+1}}{x_i}\right)\left(1 - t\frac{x_{N_0+2}}{x_i}\right) = \sum_{n,m,\gamma}(-t)^{n+m+1} c_{n,m,\gamma}\, f_{n,m,\gamma}(x^{-1})\, x_{N_0+1}^{n+1} x_{N_0+2}^{m-1}$$

Once more, $f_{n,m,\gamma}$ is a monomial in $x_1^{-1}, \ldots, x_{N_0}^{-1}$ with exponents no greater than -2 and $c_{n,m,\gamma}$ is a positive integer. The only terms in this expansion which, when multiplied by x^δ, have distinct exponents occur when *either* $1 \leq m \leq N_0$ and $m-1 \leq n \leq N_0-1$ *or* $n = N_0$, $m = 0$.

In the latter case, $c_{N_0,0,\gamma} = 1$, $f_{N_0,0,\gamma} = (x_1 x_2 \cdots x_{N_0})^{-1}$ and

$$f_{N_0,0,\gamma}\, x_{N_0+1}^{N_0+1} x_{N_0+2}^{-1} x^\delta = x^{\sigma(\alpha_2+\delta)} \qquad \operatorname{sgn}\sigma = (-1)^{N_0} \qquad (3.19)$$

In the former case the monomials $f_{n,m,\gamma}$ take one of the $n-m+1$ possible forms

$$f_{n,m,\gamma} = \begin{cases} x_{N_0-n}^{-2} x_{N_0-n+2}^{-1} x_{N_0-n+3}^{-1} \cdots x_{N_0-m+1}^{-1} x_{N_0-m+2}^{-2} \cdots x_{N_0}^{-2} \\ x_{N_0-n}^{-1} x_{N_0-n+1}^{-2} x_{N_0-n+3}^{-1} \cdots x_{N_0-m+1}^{-1} x_{N_0-m+2}^{-2} \cdots x_{N_0}^{-2} \\ \vdots \\ x_{N_0-n}^{-1} x_{N_0-n+1}^{-1} \cdots x_{N_0-m+1}^{-1} x_{N_0-m}^{-2} x_{N_0-m+2}^{-2} \cdots x_{N_0}^{-2} \end{cases} \qquad (3.20)$$

as well as the additional form

$$f_{n,m,\gamma} = x_{N_0-n}^{-1} x_{N_0-n+1}^{-1} \cdots x_{N_0-m}^{-1} x_{N_0-m+1}^{-1} x_{N_0-m+2}^{-2} \cdots x_{N_0}^{-2} \qquad (3.21)$$

For the monomials (3.20) the corresponding $c_{n,m,\gamma} = 1$, but for the monomial (3.21), $c_{n,m,\gamma} = n-m+2$. Moreover, in the former case for each $f_{n,m,\gamma}$, we have $f_{n,m,\gamma}(x^{-1}) x_{N_0+1}^{n+1} x_{N_0+2}^{m-1} x^\delta =$

$x^{\sigma(\delta)}$, where $\operatorname{sgn}\sigma = (-1)^{n+m+1}$, while for the latter, $\operatorname{sgn}\sigma = (-1)^{n+m}$. Hence, combining contributions from (3.20), (3.21) and (3.19), we get

$$\begin{aligned}\text{LHS of (3.18)} \quad = \quad & \sum_{m=1}^{N_0}\sum_{n=m-1}^{N_0-1}(-t)^{n+m+1}\Big((n-m+1)(-1)^{n+m+1}+(n-m+2)(-1)^{n+m}\Big)[x^\delta]G_0 \\ & -t^{N_0+1}[x^{\alpha_2+\delta}]G_0\end{aligned}$$

which produces the required result after summation.

Lemmas 3.5 and 3.6 show that everything on the right hand side of (3.17) can be expressed in terms of $[x^\delta]G_0$ and $[x^{\alpha_2+\delta}]G_0$. However, by eliminating $[x^{\alpha_1+\delta}]G_0$ from (3.7) and (3.8), we have

$$[x^{\alpha_2+\delta}]G_0 = \frac{1}{(1-qst^{N_0+1})}\left(\frac{1-t^{N_0+1}}{1-t}\right)\left\{(t-qs)+q(u-t)\frac{(s-1)(1-t^{N_0+2})}{(1-qut^{N_0+1})(1-t)}\right\}[x^\delta]G_0$$

Thus, from (3.17)

$$\begin{aligned}\frac{[x^0]F_1}{[x^\delta]G_0} = (1+t+t^2)\times\frac{1}{1-t}&\left(\frac{(1-t^{N_0+1})}{(1-t)}-t\frac{(1-t^{2N_0+2})}{(1-t^2)}\right)\\ -\frac{t}{1-t}&\left(t\frac{(1-t^{2N_0})}{(1-t^2)}-t^{N_0+1}\frac{(1-t^{N_0})}{(1-t)}\right)\\ -t\frac{1}{(1-qst^{N_0+1})}\left(\frac{1-t^{N_0+1}}{1-t}\right)&\left\{(t-qs)+q(u-t)\frac{(s-1)(1-t^{N_0+2})}{(1-qut^{N_0+1})(1-t)}\right\}\end{aligned}$$

Simplification of this expression yields the desired result (3.16).

4. CONJECTURES INVOLVING MACDONALD POLYNOMIALS

4.1 A Gram-Schmidt construction

The q-generalization (2.10) of (1.1) can be used as a weight function in the Gram-Schmidt construction of the q-generalization of the polynomials satisfying conditons (i) and (ii) of Section 1. Thus define the q-generalization of the inner product (1.2) by

$$\begin{aligned}&\langle f|g\rangle_{N_0,\dots,N_p;\lambda;q}\\ &:= \prod_{l=1}^{N_0}\int_{-1/2}^{1/2}dy_l\,\prod_{\alpha=1}^{p}\prod_{l=1}^{N_\alpha}\int_{-1/2}^{1/2}dx_l^{(\alpha)}\,|\psi_0(\{z_j^{(\alpha)}\}_{\substack{\alpha=1,\dots,p\\ j=1,\dots,N_\alpha}},\{w_j\}_{j=1,\dots,N_0};q)|^2 f^*\,g. \qquad (4.1)\end{aligned}$$

and define the q-generalization of the symmetric polynomials p_κ, $p_\kappa(w_1,\dots,w_{N_0};q)$ say, by properties (i) and (ii) of Section 1 with the inner product therein replaced by (4.1). Note from condition (i) that

$$p_{1^k}(w_1,\dots,w_{N_0};q) = m_{1^k} := \sum_{1\le j_1<\dots<j_k\le N_0} w_{j_1}w_{j_2}\dots w_{j_k}. \qquad (4.2)$$

Based on some exact computer generated data, and the conjecture (1.3) for the $q=1$ case, we make the following conjecture.

Conjecture 4.1 We have

$$p_\kappa(w_1,\dots,w_{N_0};q) = P_\kappa(w_1,\dots,w_{N_0};qt^p,t), \qquad \text{where} \quad t := q^\lambda \qquad (4.3)$$

and P_κ denotes the Macdonald polynomial. In the limit $q \to 1$, Conjecture 4.1 reduces to (1.5). Note in particular that this agrees with the original conjecture [7, Conj. 2.4] only in the case $p=1$.

We have obtained exact computer generated data for two further conjectures which are closely related to this result. In relation to the first conjecture we note from Conjecture 4.1 that p_κ is independent of $N_1,\dots,N_p$. Analogous to the $q=1$ case [7, Conj. 2.3], this can be understood in terms of a conjecture which generalizes Conjecture 2.2.

Conjecture 4.2 Let $h = h(w_1,\dots,w_{N_0})$ be a Laurent polynomial of the form $h = \sum_{\bar\sigma \le \bar\rho} c_\sigma m_\sigma$, where $\rho = (\rho_1,\dots,\rho_{N_0})$, $|\rho_1| \ge \dots \ge |\rho_{N_0}|$ and $\bar\rho = (|\rho_1|,\dots,|\rho_{N_0}|)$, and let

$$D_p(N_1,\dots,N_p;N_0;a,b,\lambda;q)[h] := \langle h | A(\{z_j^{(\alpha)}\}_{\substack{\alpha=1,\dots,p\\ j=1,\dots,N_\alpha}}, \{w_j\}_{j=1,\dots,N_0};q)\rangle_{N_0,\dots,N_p;\lambda;q}$$

where $A(\{z_j^{(\alpha)}\},\{w_j\ q)$ is given by (2.16b). For $N_p \ge N_j - 1$ and $N_j \ge |\rho_1| - 1$ $(j=1,\dots,p-1)$ we conjecture that

$$\frac{D_p(N_1,\dots,N_{p-1},N_p+1;N_0;a,b,\lambda;q)[h]}{D_p(N_1,\dots,N_{p-1},N_p;N_0;a,b,\lambda;q)[h]}$$

is given by the r.h.s. of Conjecture 2.2.

From Conjecture 4.2 it follows that for $N_p \ge N_j - 1$ and $N_j \ge |\rho_1| - 1$ $(j=1,\dots,p-1)$

$$D_p(N_1,\dots,N_p;N_0;0,0,\lambda;q)[h] = f_{p-1}(N_1,\dots,N_{p-1};N_0;\lambda;q)[h] A_{p,|\rho_1|}(N_1,\dots,N_p;N_0;\lambda;q) \tag{4.4}$$

where $A_{p,|\rho_1|}$ denotes the r.h.s. of (2.17) with N_p replaced by l and the product formed over l from $l = |\rho_1| - 1$ to $N_p - 1$. From the symmetry

$$D_p(N_1,\dots,N_{p-2},N,N-1;N_0;0,0,\lambda;q)[h] = D_p(N_1,\dots,N_{p-2},N-1,N;N_0;0,0,\lambda;q)[h]$$

we see from (4.4) that for some function f_{p-2} which is independent of N_p and N_{p-1}

$$f_{p-1}(N_1,\dots,N_{p-1};N_0;\lambda;q)[h] = A_{p-1,|\rho_1|}(N_1,\dots,N_{p-1};N_0;\lambda;q) f_{p-2}(N_1,\dots,N_{p-2};N_0;\lambda;q)[h]$$

where

$$A_{p-1,|\rho_1|}(N_1,\dots,N_{p-1};N_0;\lambda;q) := \prod_{n=|\rho_1|}^{N_{p-1}} \frac{A_{p,|\rho_1|}(N_1,\dots,N_{p-2},n-1,n;N_0;\lambda;q)}{A_{p,|\rho_1|}(N_1,\dots,N_{p-2},n,n-1;N_0;\lambda;q)}.$$

Thus

$$\begin{aligned} &D_p(N_1,\dots,N_p;N_0;0,0,\lambda;q)[h] = \\ &\quad f_{p-2}(N_1,\dots,N_{p-2};N_0;\lambda;q)[h] A_{p,|\rho_1|}(N_1,\dots,N_p;N_0;\lambda;q) A_{p-1,|\rho_1|}(N_1,\dots,N_{p-1};N_0;\lambda;q). \end{aligned}$$

Proceeding in this fashion we see that the dependence of D_p on $N_1,\dots,N_p$ factorizes from the dependence on h, N_0 and p and thus cancels out of the ratio of inner products which define the coefficients in the Gram-Schmidt procedure.

In relation to the second conjecture, we recall [12] that the Macdonald polynomial $P_\kappa(w_1,\dots,w_{N_0};q,t)$ is an eigenfunction of the (mutually commuting) operators

$$M_{N_0}^{(r)}(q,t) := \sum_I A_I(w;t) \prod_{i\in I} T_{q,w_i}$$

summed over all r-element subsets of $\{1, 2, \ldots, N_0\}$, where

$$A_I(w;t) = t^{r(r-1)/2} \left(\prod_{\substack{i \in I \\ j \notin I}} \frac{tw_i - w_j}{w_i - w_j} \right) \tag{4.5a}$$

and T_{q,w_i} is the q-shift operator with action

$$T_{q,w_i} f(w_1, \ldots, w_{i-1}, w_i, w_{i+1}, \ldots, w_{N_0}) = f(w_1, \ldots, w_{i-1}, qw_i, w_{i+1}, \ldots, w_{N_0}). \tag{4.5b}$$

Furthermore, the fact that the set of Macdonald polynomials $\{P_\kappa(w_1, \ldots, w_{N_0}; q, q^\lambda\}_\kappa$ are orthogonal with respect to the inner product (4.1) with $p = 0$ follows from the eigenfunction property and the fact that $M_{N_0}^{(r)}(q,t)$ is Hermitian with respect to this inner product:

$$\langle M_{N_0}^{(r)}(q,t) f | g \rangle_{N_0;\lambda;q} = \langle f | M_{N_0}^{(r)}(q,t) g \rangle_{N_0;\lambda;q} \tag{4.6}$$

For general p we have obtained exact computer-generated data which suggest a result similar to (4.6).

Conjecture 4.3 Let m_κ and m_μ be monomial symmetric functions and suppose

$$N_j \geq \begin{cases} \min(\kappa_1, \mu_1), & \kappa_1 \neq \mu_1 \\ \kappa_1 - 1, & \kappa_1 = \mu_1 \end{cases}$$

$(j = 1, \ldots, p)$. We have

$$\langle M_{N_0}^{(r)}(qt^p, t) m_\kappa | m_\mu \rangle_{N_0, \ldots, N_p; \lambda; q} = \langle m_\kappa | M_{N_0}^{(r)}(qt^p, t) m_\mu \rangle_{N_0, \ldots, N_p; \lambda; q}.$$

Remarks

1. Since the set of functions $\{P_\kappa(w_1, \ldots, w_{N_0}; qt^p, t\}_\kappa$ are eigenfunctions of $M_{N_0}^{(r)}(qt^p, t)$ with distinct eigenvalues, we see that Conjecture 4.1 follows as a corollary of Conjecture 4.3.

2. As $q \to 1$ Conjecture 4.3 implies that the Laplace-Beltrami operator

$$D_{N_0}(p + 1/\lambda) = -\frac{(p + 1/\lambda)^2}{2} \sum_{i=1}^{N_0} w_i \frac{\partial^2}{\partial w_i^2} - (p + 1/\lambda) \sum_{i \neq j} \frac{w_i^2}{w_i - w_j} \frac{\partial}{\partial w_i},$$

which is (up to a shift by a constant) the limit $q \to 1$ of

$$\frac{1}{(q^\lambda - 1)^2} \left(M_{N_0}^{(2)} - (N_0 - 1)\, M_{N_0}^{(1)} + \frac{1}{2} N_0 (N_0 - 1)\, M_{N_0}^{(0)} \right),$$

is Hermitian with repect to the multi-component Jack inner product (1.2), given the same constraints on $N_1, \ldots, N_p$.

4.2 Normalization integral

In the $q = 1$ case the normalization of (1.3) with respect to the inner product (1.2) has been conjectured in ref.[2, Section 3]. For general q the corresponding normalization is given by the inner product

$$\langle P_\kappa(w_1, \ldots, w_{N_0}; qt, t) | P_\kappa(w_1, \ldots, w_{N_0}; qt, t) \rangle_{N_0, \ldots, N_p; \lambda; q} =: \mathcal{N}_p^\kappa(N_1, \ldots, N_p; N_0; \lambda; q) \tag{4.7}$$

with κ_1 restricted as in condition (i) of Section 1. As a corollary to Conjecture 4.2, the dependence on $N_1, \dots, N_p$ factorizes from the dependence on κ, N_0 and p, so it suffices to consider the case $N_1 = \dots = N_p = \kappa_1$. Explictly, in the case $p = 1$ Conjecture 4.2 gives

$$\mathcal{N}_1^{\kappa}(N_1; N_0; \lambda; q) = \mathcal{N}_1^{\kappa}(\kappa_1; N_0; \lambda; q) \prod_{j=\kappa_1}^{N_1-1} \frac{[(\lambda+1)(j+1)]_q}{\Gamma_q(1+\lambda)} \Big((\lambda+1)j + \lambda N_0 + 1; q\Big)_\lambda \tag{4.8}$$

To obtain a conjecture for the evaluation of $\mathcal{N}_1^{\kappa}(N_1; N_0; \lambda; q)$ we have obtained some exact computer generated data, as well as an analytic result in the case $\kappa = 1^k$, $\lambda = 1$ (see the next section). These results, and the corresponding conjecture [7, Conjecture 3.1] in the $q = 1$ case, suggest a closed form expression for general κ.

Conjecture 4.4 Let f_j denote the frequency of the integer j in the partition κ so that $\kappa = \kappa_1^{f_{\kappa_1}} (\kappa_1 - 1)^{f_{\kappa_1 - 1}} \dots 1^{f_1}$. We have

$$\mathcal{N}_1^{f_{\kappa_1} f_{\kappa_1-1} \dots f_1}(\kappa_1; N_0; \lambda; q) = \frac{[\kappa_1]_q! \Gamma_{q^\lambda}(\lambda N_0 + 1)}{(\Gamma_q(1+\lambda))^{N_0+\kappa_1}} \prod_{j=1}^{\kappa_1} \Big(\lambda f_j + 1; q\Big)_\lambda$$

$$\times \prod_{j=1}^{\kappa_1} \frac{\Big((\lambda+1)j + 1 + \lambda \sum_{k=1}^{\kappa_1+1-j} f_{j+k-1}; q\Big)_\lambda \Big((\lambda+1)(j-1) + 1 + \lambda(N_0 - \sum_{k=1}^{\kappa_1+1-j} f_{j+k-1}); q\Big)_\lambda}{\Big((\lambda+1)j + 1 + \lambda f_j; q\Big)_\lambda} .$$

5. AN EXPLICIT GRAM-SCHMIDT CONSTRUCTION AND COMPUTATION OF THE NORMALIZATION

5.1 A q-determinant method

In ref. [7, Proposition 2.1] a determinant method was used to prove, for $q = 1$, Conjecture 4.1 in the case $p = \lambda = 1$, $\kappa = 21^k$ and Conjecture 4.4 in the case $p = \lambda = 1$. It is possible to q-generalize the determinant method and thus prove these results for general q.

Proposition 5.1 For $p = \lambda = 1$ and $N_1 \geq 1$ we have

$$\langle m_{1^{k+2}} | m_{1^{k+2}} \rangle_{N_0; N_1; 1; q} = [N_0]_q! [N_1]_{q^2}! [k+3]_q [N_0 - 1 - k]_q \prod_{l=1}^{N_1-1} [N_0 + 2N_1 + 1 - 2l]_q \tag{5.1}$$

and

$$\langle m_{1^{k+2}} | s_{21^k} \rangle_{N_0; N_1; 1; q} = q [N_0]_q! [N_1]_{q^2}! [k+1]_q [N_0 - 1 - k]_q \prod_{l=1}^{N_1-1} [N_0 + 2N_1 + 1 - 2l]_q \tag{5.2}$$

where $s_\kappa = s_\kappa(w_1, \dots, w_{N_0})$ denote the Schur polynomial. Consequently

$$p_{21^k}(w_1, \dots, w_{N_0}; q) = s_{21^k} - q \frac{[k+1]_q}{[k+3]_q} s_{1^{k+2}}. \tag{5.3}$$

Proof We will first show how to deduce (5.3) from (5.1) and (5.2). Now, since the Schur polynomials can be written

$$s_\kappa = m_\kappa + \sum_{\mu < \kappa} a'_\mu m_\mu$$

for some coefficients a'_μ, the condition (i) (recall Section 1) in the definition of p_κ can be rewritten as

$$p_\kappa(w_1,\dots,w_{N_0};q) = s_\kappa + \sum_{\mu<\kappa} a_\mu s_\mu.$$

Furthermore,

$$s_{1^n} = m_{1^n} = p_{1^n}(w_1,\dots,w_{N_0};q),$$

so by the Gram-Schmidt procedure

$$p_{21^k}(w_1,\dots,w_{N_0};q) = s_{21^k}(w_1,\dots,w_{N_0}) - \frac{\langle m_{1^{k+2}}|s_{21^k}\rangle_{N_0;N_1;1;q}}{\langle m_{1^{k+2}}|m_{1^{k+2}}\rangle_{N_0;N_1;1;q}} s_{1^{k+2}}(w_1,\dots,w_{N_0}) \tag{5.4}$$

Substitution of (5.1) and (5.2) into (5.4) gives (5.3).

Next we take up the task of deriving the results (5.1) and (5.2). We first transform the integrand into a form symmetric in $\{z_j\}$ and $\{w_j\}$ by appealing to a lemma of Kadell [9, lemma 4], which for any f and g symmetric in $\{z_j\}$ and $\{w_j\}$ gives the identity

$$\langle g|f\rangle_{N_0;N_1;1;q} = \frac{[N_0]_q!}{N_0!}\frac{[N_1]_{q^2}!}{N_1!}\prod_{l=1}^{N_1}\int_{-1/2}^{1/2} dx_l \prod_{l=1}^{N_0}\int_{-1/2}^{1/2} dy_l\, F(\{z_j\},\{w_j\};q) f(\{z_j\},\{w_j\};q)$$
$$\times g(\{z_j^*\},\{w_j^*\};q)$$

where

$$F(\{z_j\},\{w_j\};q) = F_1(\{w_j\})F_2(\{z_j\},\{w_j\};q)$$

with

$$F_1 = \prod_{1\le j<k\le N_0}(w_k - w_j),$$

$$F_2 = \prod_{1\le j<k\le N_0}(w_k^{-1} - w_j^{-1})$$
$$\times \prod_{j=1}^{N_0}\prod_{\alpha=1}^{N_1}\Big(1-\frac{z_\alpha}{w_j}\Big)\Big(1-q\frac{w_j}{z_\alpha}\Big)\prod_{1\le\alpha<\beta\le N_1}\Big(1-\frac{z_\beta}{z_\alpha}\Big)\Big(1-\frac{z_\alpha}{z_\beta}\Big)\Big(1-q\frac{z_\beta}{z_\alpha}\Big)\Big(1-q\frac{z_\alpha}{z_\beta}\Big)$$

(this factorization of F is chosen for later convenience). Thus

$$\langle m_{1^{k+2}}|m_{1^{k+2}}\rangle_{N_0;N_1;1;q} = \frac{[N_0]_q!}{N_0!}\frac{[N_1]_{q^2}}{N_1!}$$
$$\times \prod_{l=1}^{N_1}\int_{-1/2}^{1/2} dx_l \prod_{l=1}^{N_0}\int_{-1/2}^{1/2} dy_l F(\{z_j\},\{w_j\};q) m_{1^{k+2}}(\{w_j\}) m_{1^{k+2}}(\{w_j^*\}) \tag{5.5}$$

and similarly the inner product in (5.2).

Consider now the task of evaluating the integral in (5.5). Our method is to write F in terms of determinants. From the Vandermonde determinant identity

$$\prod_{1\le j<k\le N_0+2N_1}(u_k-u_j) = \det[u_j^{k-1}]_{j,k=1,\dots,N_0+2N_1},$$

with

$$u_j = w_j^{-1}\ (j=1,\dots,N_0)\quad u_{j+N_0} = z_j^{-1}\ (j=1,\dots,N_1)\quad u_{j+N_0+N_1} = qz_j^{-1}\ (j=1,\dots,N_1),$$

straightforward manipulation gives

$$F_2(\{z_j\},\{w_j\};q) = (-1)^{N_1+N_0N_1}(1-q)^{-N_1}q^{-N_1(N_1-1)/2}\prod_{j=1}^{N_1} z_j^{1+2(N_1-1)+N_0}\prod_{j=1}^{N_0} w_j^{N_1}$$

$$\times \det\begin{bmatrix} [w_j^{-(l-1)}]_{\substack{j=1,\dots,N_0 \\ l=1,\dots,N_0+2N_1}} \\ \begin{bmatrix} z_j^{-(l-1)} \\ (qz_j^{-1})^{(l-1)} \end{bmatrix}_{\substack{j=1,\dots,N_1 \\ l=1,\dots,N_0+2N_1}} \end{bmatrix} \tag{5.6}$$

(the block notation in (5.6) indicates successive rows; thus the row with elements $z_1^{-(l-1)}$ is followed by $(qz_1^{-1})^{l-1}$, which is followed by $z_2^{-(l-1)}$ etc.). Also, since $m_{1^{k+2}} = s_{1^{k+2}}$, from the determinant formula for the Schur polynomials we have

$$F_1 s_{1^{k+2}} = \det[w_j^{l+\kappa_{N_0-j+1}-1}]_{j,l=1,\dots,N_0} \tag{5.7}$$

where $\kappa_j = 1$ $(j=1,\dots,k+2)$, $\kappa_j = 0$ otherwise.

Since (5.6) and (5.7) are antisymmetric with respect to interchanges of $w_1,\dots,w_{N_0}$, in the integral (5.5) we can replace (5.7) by $N_0!$ times its diagonal term. In the definition of $m_{1^{k+2}}(\{w_j^*\})$:

$$m_{1^{k+2}}(\{w_j^*\}) = \sum_{1\le j_1<\dots<j_{k+2}\le N_0} w_{j_1}^{-1}w_{j_2}^{-1}\dots w_{j_{k+2}}^{-1} \tag{5.8}$$

take the sum outside the integral and multiply all terms from the summand of (5.8) and the diagonal term of the determinant (5.7) into appropriate rows of (5.6). Row-by-row integration of the determinant with respect to $w_1,\dots,w_{N_0}$ gives a non-zero contribution in row j only in column

$$l = N_1 + j + \kappa_{N_0-j+1} - \xi_j, \qquad \xi_j := \begin{cases} 1 & \text{if} \quad j = j_1,\dots,j_{k+2} \\ 0 & \text{otherwise} \end{cases} \tag{5.9a}$$

and this term is equal to unity. For these non-zero columns to be distinct and the determinant thus non-zero we require

$$\{j_1,\dots,j_{k+2}\} = \{1,\dots,\nu,N_0-k-1,\dots,N_0-\nu\}. \tag{5.9b}$$

for some $\nu = 0,\dots,k+2$. Assuming this condition and expanding the integrated determinant by the non-zero columns gives, after expanding the remaining terms and grouping in pairs

$$\begin{aligned} &\langle m_{1^{k+2}}|m_{1^{k+2}}\rangle_{N_0;N_1;1;q} \\ &= \frac{[N_0]_q![N_1]_{q^2}}{N_1!}(1-q)^{-N_1}q^{-N_1(N_1-1)/2}\sum_{P(2\alpha)>P(2\alpha-1)}\epsilon(P)\prod_{\alpha=1}^{N_1}(q^{P(2\alpha)-1}-q^{P(2\alpha-1)-1}) \\ &\quad\times\int_{-1/2}^{1/2}dx_\alpha\, z_\alpha^{N_0+2N_1+1-P(2\alpha)-P(2\alpha-1)} \end{aligned} \tag{5.10a}$$

where

$$P(\alpha)\in\{1,\dots,N_1-1\}\cup\{N_1+\nu\}\cup\{N_1+N_0-\nu+1\}\cup\{N_1+N_0+2,\dots,N_0+2N_1\}. \tag{5.10b}$$

A non-zero contribution to (5.9) requires

$$P(2\alpha-1) = N_0+2N_1+1-P(2\alpha) \tag{5.11a}$$

and

$$P(2\alpha) \in \{N_1 + N_0 + 1 - \nu\} \cup \{N_1 + N_0 + 2, \dots, 2N_1 + N_0\}. \tag{5.11b}$$

Each of the $N_1!$ different choices (5.11b) give the same contribution to (5.9) and so

$$\langle m_{1^{k+2}} | m_{1^{k+2}} \rangle_{N_0;N_1;1;q}$$
$$= [N_0]_q! [N_1]_{q^2}! (1-q)^{-N_1} q^{-N_1(N_1-1)/2} \prod_{l=1}^{N_1-1} (q^{N_0+2N_1-l} - q^{l-1}) \sum_{\nu=0}^{k+2} (q^{N_0+N_1-\nu} - q^{N_1+\nu-1}) \tag{5.12}$$

which after straightforward simplification gives (5.1).

The computation of the analogue of (5.5) for the inner product (5.2) is very similar. In place of (5.7) we have

$$F_1 s_{21^k}(\{w_j\}) = \det[w_j^{l+\kappa_{N_0-j+1}-1}]_{j,l=1,\dots,N_0} \tag{5.13}$$

where $\kappa_1 = 2$, $\kappa_j = 1$ ($j = 2, \dots, k+1$), $\kappa_j = 0$ otherwise. After integration over $w_1, \dots, w_{N_0}$ the condition (5.9a) still gives the column number of the non-zero entry in row j. For the non-zero columns to be distinct and the condition (5.11a) to hold we see that in place of (5.9b) we require

$$\{j_1, \dots, j_{k+2}\} = \{1, \dots, \nu, N_0 - k, \dots, N_0 - \nu + 1\}$$

for some $\nu = 1, \dots, k+1$. This shows that $\langle m_{1^{k+2}} | s_{21^k} \rangle_{N_0;N_1;1;q}$ is given by the r.h.s. of (5.10a) with condition (5.10b), which simplifies down to the r.h.s. of (5.12), the only difference being that the summation over ν is now from $\nu = 1$ to $k+1$. After evaluating the sum the result (5.2) follows.

Proposition 5.1 immediately establishes Conjecture 4.4 in the case $p = \lambda = 1$. To prove Conjecture 4.1 in the case $p = \lambda = 1$, $\kappa = 21^k$ it is necessary to identify (5.3) as the corresponding Macdonald polynomial. In the $q = 1$ case this can be done by appealing to a theorem of Stanley [14, Prop. 7.2] which gives the explicit expansion of $J^{(\alpha)}_{2^p 1^q}$ in terms of monomial symmetric functions. As we know of no corresponding result for the Macdonald polynomials, it remains to show that $P_{21^k}(w_1, \dots, w_{N_0}; q^2, q)$ is given by the r.h.s. of (5.3). This can be done by the characterisation of the Macdonald polynomial as an eigenfunction of the operator (4.5a) with $r = 1$.

5.2 Expansion of $P_{21^k}(w_1, \dots, w_{N_0}; q^2, q)$ in terms of Schur polynomials

We know that $P_{21^{n-2}} \equiv P_{21^{n-2}}(w_1, \dots, w_{N_0}; q, t)$ must have the form

$$P_{21^{n-2}} = s_{21^{n-2}} + \gamma\, s_{1^n} = m_{21^{n-2}} + (\gamma + n - 1)\, m_{1^n} \tag{5.14}$$

where γ is to be determined. The action of the operator $M^{(1)}_{N_0}(q,t)$ (see (4.4a)), on the monomial symmetric functions is given explicitly by[12]

$$M^{(1)}_{N_0}(q,t)\, m_\kappa = \sum_\alpha \sum_{i=1}^{N_0} t^{N_0-i} q^{\alpha_i}\, s_\alpha \tag{5.15}$$

where the outer sum is over all derangements $\alpha \in \mathbb{N}^{N_0}$ of the partition κ. Due to the modification rules for Schur functions associated with unordered partitions (if $\kappa_i < \kappa_{i+1}$ for any i, then $s_{(\dots,\kappa_i,\kappa_{i+1},\dots)} = -s_{(\dots,\kappa_{i+1}-1,\kappa_i+1,\dots)}$; in particular $s_{(\dots,\kappa_i,\kappa_i+1,\dots)} = 0$), the only distinct permutations α of $\kappa = (2, 1^{n-2}, 0^{N_0-n+1})$ for which s_α is non-zero are of the form $(2, 1^{n-2}, 0^{N_0-n+1})$, or $(1^p, 0, 2, 1^{n-2-p}, 0^{N_0-n})$, $p = 0, 1, \dots, n-2$. It thus follows from (5.15) that

$$M^{(1)}_{N_0}(q,t)\, m_{21^{n-2}} = A_1\, s_{21^{n-2}} + A_2\, s_{1^n} \tag{5.16}$$

where

$$\begin{aligned} A_1 &= q^2 t^{N_0-1} + q t^{N_0-n+1}[n-2]_t + [N_0-n+1]_t \\ A_2 &= -q^2 t^{N_0-n}[n-1]_t - q\left((n-2)(t^{N_0-n}+t^{N_0-1}) + (n-3)t^{N_0-n+1}[n-2]_t\right) \\ &\quad -t^{N_0-n+1}[n-1]_t - (n-1)[N_0-n]_t \end{aligned}$$

Recalling that $P_{1^n}(w;q,t) \equiv m_{1^n}(w)$ (and hence m_{1^n} is an eigenfunction of $M_{N_0}^{(1)}(q,t)$) it follows from (5.14) and (5.16) that

$$M_{N_0}^{(1)}(q,t)\, P_{21^{n-2}} = A_1\, s_{21^{n-2}} + (A_2 + (\gamma+n-1)e(1^n))\; s_{1^n}$$

where $e(\kappa) := \sum_{i=1}^{N_0} t^{N_0-i} q^{\kappa_i}$ is the eigenvalue of $P_\kappa(w;q,t)$ under $M_{N_0}^{(1)}(q,t)$. However

$$M_{N_0}^{(1)}(q,t)\, P_{21^{n-2}} = e(21^{n-2})\, P_{21^{n-2}} = e(21^{n-2})\,(s_{21^{n-2}} + \gamma s_{1^n})$$

Equating the coefficients of s_{1^n} in these two equations (which is permissible since the set of Schur functions s_κ, for κ a proper partition, is linearly independent) yields

$$\gamma = \frac{A_2 + (n-1)e(1^n)}{e(21^{n-2}) - e(1^n)} = \frac{(q-t)}{(1-qt^{n-1})}\,[n-1]_t$$

In the particular case $(q,t) \to (q^2,q)$ and $n = k+2$, this reproduces the r.h.s. of (5.3) and thus completes the proof of Conjecture 4.1 in the case $p = \lambda = 1$, $\kappa = 21^k$.

Appendix

In this appendix an application of a recently derived extension of the q-Morris constant term identity involving the Macdonald polynomial will be given. The extension is [10, Theorem 4]

$$\begin{aligned} &\mathrm{CT}\, P_\kappa(t_1,\dots,t_n;q,q^\lambda) \prod_{i=1}^n (t_i;q)_a \Big(\frac{q}{t_i};q\Big)_b \prod_{1\le i<j\le n} \Big(\frac{t_i}{t_j};q\Big)_\lambda \Big(q\frac{t_j}{t_i};q\Big)_\lambda \\ &= (-1)^{|\kappa|} q^{\sum_{i=1}^n \kappa_i(\kappa_i+1)/2} \\ &\quad \times \frac{(q^\lambda;q^\lambda)_n}{(1-q^\lambda)^n} \prod_{1\le i<j\le n} (q^{\kappa_i-\kappa_j} q^{\lambda(j-i)};q)_\lambda \prod_{i=1}^n \frac{(q;q)_{a+b+(n-i)\lambda}}{(q;q)_{a+(n-i)\lambda+\kappa_i}(q;q)_{b+(i-1)\lambda-\kappa_i}} \\ &= q^{(b+1)|\kappa|} D_0(n;a,b,\lambda;q) P_\kappa(1,q^\lambda,\dots,q^{(n-1)\lambda};q,q^\lambda) \frac{[-b]_{\kappa;q}^{(1/\lambda)}}{[a+1+(n-1)\lambda]_{\kappa;q}^{(1/\lambda)}} \qquad \text{(A1)} \end{aligned}$$

where a,b,λ are assumed to be non-negative integers, $D_0(n;a,b,\lambda;q)$ is given by (2.14) with $N_1 = 0$, $N_0 = n$,

$$\begin{aligned} [x]_{\kappa;q}^{(1/\lambda)} &:= \prod_{j=1}^n \frac{\Gamma_q(x-\lambda(j-1)+\kappa_j)}{\Gamma_q(x-\lambda(j-1))} \qquad \text{(A2)} \\ &= \prod_{j=1}^n [x-\lambda(j-1)+\kappa_j-1]_q \dots [x-\lambda(j-1)]_q \qquad \text{(A2)} \end{aligned}$$

and we have used the formula [12]

$$P_\kappa(1,q^\lambda,\dots,q^{(n-1)\lambda};q,q^\lambda) = q^{\lambda\sum_{i=1}^n (i-1)\kappa_i} \prod_{1\le i<j\le n} \frac{(q^{\kappa_i-\kappa_j} q^{\lambda(j-i)};q)_\lambda}{(q^{\lambda(j-i)};q)_\lambda}$$

and the manipulation

$$\frac{(q;q)_p}{(q;q)_{p-\kappa_i}} = (-1)^{\kappa_i} q^{p\kappa_i} q^{-\kappa_i(\kappa_i-1)/2}[-p+\kappa_i-1]_q \dots [-p]_q (1-q)^{\kappa_i}$$

with $p = b + (i-1)\lambda$. Using (A1) we can calculate the expansion of the power sums in terms of Macdonald polynomials. We will require a simple lemma, which was used in a special case in ref. [8, Proposition 2].

Proposition A1 Let $f(w_1, \dots, w_n)$ be symmetric in $w_1, \dots, w_n$ ($w_j = e^{2\pi i y_j}$), periodic of period 1 in each variable y_j and homogenoeous of integer order k ($k \ne 0$). Let $u_\epsilon(w_l)$ have the small-ϵ expansion

$$u_\epsilon(w_l) = 1 + \epsilon a(w_l) + O(\epsilon^2).$$

We have

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \prod_{l=1}^n \int_{-1/2}^{1/2} dy_l\, u_\epsilon(w_l)\, f(w_1, \dots, w_n) = n \int_{-1/2}^{1/2} u_\epsilon(w_1)\, dy_1 \prod_{l=2}^n \int_{-1/2}^{1/2} dy_l\, f(1, w_2, \dots, w_n).$$

Proof For small-ϵ

$$\prod_{l=1}^n a_\epsilon(w_l) \ \sim\ 1 + \epsilon \sum_{l=1}^n a(w_l),$$

and thus, since f is assumed homogeneous of non-zero integer order

$$\prod_{l=1}^n \int_{-1/2}^{1/2} dy_l\, u_\epsilon(w_l)\, f(w_1, \dots, w_n) \ \sim\ \epsilon \prod_{l=1}^n \int_{-1/2}^{1/2} dy_l \sum_{l=1}^n a(w_l) f(w_1, \dots, w_n).$$

The stated result now follows by using the assumption that f is symmetric to replace $\sum_{l=1}^n a(w_l)$ in the integrand by $na(w_1)$, then using the assumption that f is periodic to replace w_j by $w_1 w_j$ ($j = 2, \dots, n$) and finally the fact that f is homogeneous of order k to write

$$f(w_1, w_1 w_2 \dots, w_1 w_n) = w_1^k f(1, w_2, \dots, w_n).$$

The expansion of the power sums is given by the following result.

Proposition A2 For $k \in Z_{>0}$ we have

$$\sum_{i=1}^n w_i^k = \sum_{|\kappa|=k} \frac{\alpha_{\kappa;q}}{\langle P_\kappa | P_\kappa \rangle'} P_\kappa(w_1, \dots, w_n; q, q^\lambda)$$

where

$$\alpha_{\kappa;q} = \frac{[|\kappa|]_q \Gamma_q(\kappa_1)}{[n]_{q^\lambda}!} \frac{\Gamma_q(\lambda n + 1)}{(\Gamma_q(\lambda+1))^n} P_\kappa(1, q^\lambda, \dots q^{(n-1)\lambda}) \frac{[0]'^{(1/\lambda)}_{\kappa;q}}{[1+(n-1)\lambda]^{(1/\lambda)}_{\kappa;q}}$$

(the dash on $[0]'^{(1/\lambda)}_{\kappa;q}$ means that the $j=1$ term in its definition (A2) is to be omitted)and

$$\begin{aligned}
\langle P_\kappa | P_\kappa \rangle' &:= \frac{1}{n!} \mathrm{CT}\, P_\kappa(t_1, \dots, t_n; q, q^\lambda) P_\kappa(1/t_1, \dots, 1/t_n; q, q^\lambda) \prod_{1 \le i < j \le n} \Big(\frac{t_i}{t_j}; q\Big)_\lambda \Big(\frac{t_j}{t_i}; q\Big)_\lambda \\
&= \prod_{1 \le i < j \le n} \frac{(q^{\kappa_i - \kappa_j + \lambda(j-i)}; q)_\infty (q^{\kappa_i - \kappa_j + 1 + \lambda(j-i)}; q)_\infty}{(q^{\kappa_i - \kappa_j + \lambda(j-i+1)}; q)_\infty (q^{\kappa_i - \kappa_j + 1 + \lambda(j-i-1)}; q)_\infty}
\end{aligned}$$

(the final equality is given in ref. [12]; see also [4]).

Proof

First we write the l.h.s. of (A1) in symmetric form using the lemma of Kadell used in the proof of Proposition 5.1, and then extend its validity to general a, b, λ by using the integral (2.9) in place of the constant term and interpreting $(x;q)_a$ etc. according to (2.7). This gives

$$\prod_{l=1}^{n} \int_{-1/2}^{1/2} dy_l \, P_\kappa(w_1,\dots,w_n;q,q^\lambda) \prod_{i=1}^{n} (w_i;q)_a \Big(\frac{q}{w_i};q\Big)_b \prod_{1\le i<j\le n} \Big(\frac{w_i}{w_j};q\Big)_\lambda \Big(\frac{w_j}{w_i};q\Big)_\lambda$$

$$= \frac{n!}{[n]_{q^\lambda}!} \times (\text{final equality in (A1)}).$$

We now choose $a = 0$, $b = \epsilon$ and apply Proposition A1 with

$$u_\epsilon(w_l) = (\frac{q}{w_l};q)_\epsilon$$

and

$$f(w_1,\dots,w_n) = P_\kappa(w_1,\dots,w_n;q,q^\lambda) \prod_{1\le i<j\le n} \Big(\frac{w_i}{w_j};q\Big)_\lambda \Big(\frac{w_j}{w_i};q\Big)_\lambda \tag{A3}$$

(f is homogeneous of order $|\kappa|$). This gives

$$\prod_{l=2}^{n} \int_{-1/2}^{1/2} dy_l \, f(1,w_2,\dots,w_n)$$

$$= \frac{[|\kappa|]_q \Gamma_q(\kappa_1)}{n} \frac{n!}{[n]_{q^\lambda}!} \frac{\Gamma_q(\lambda n+1)}{(\Gamma_q(\lambda+1)^n} P_\kappa(1,q^\lambda,\dots q^{(n-1)\lambda}) \frac{[0]'^{(1/\lambda)}_{\kappa;q}}{[1+(n-1)\lambda]^{(1/\lambda)}_{\kappa;q}}, \tag{A4}$$

where we have used the formulas

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} [-\epsilon]^{(1/\lambda)}_{\kappa;q} = \log q \frac{\Gamma_q(\kappa_1)}{1-q} [0]'^{(1/\lambda)}_{\kappa;q}$$

and

$$\int_{-1/2}^{1/2} dy_l \Big(\frac{q}{w_l},q\Big)_\epsilon w_l^{|\kappa|} \sim \epsilon \log q \frac{q^{|\kappa|}}{1-q^{|\kappa|}}$$

and we have used the fact that $D_0(0,0,\lambda;q)$ is given by (2.5) with $N_0 = n$.

We remark that the formula (A4) is of direct relevance to the calculation of correlation functions in the so-called relativistic Calogero-Sutherland model [13] (the calculation of these correlations has been announced by Konno [11]). It provides the Fourier coefficients in the expansion of a symmetric sum of Dirac delta functions in terms of Macdonald polynomials. To see this we note that

$$\{P_\kappa(w_1,\dots,w_n;q;q^\lambda)\}_\kappa \bigcup \{\prod_{j=1}^{n} w_j^{-l} P_\kappa(w_1,\dots,w_n;q;q^\lambda)\}_{\kappa:\kappa_n=0,\, l=1,2,\dots}$$

form a complete set of functions which are orthogonal with respect to the inner product

$$\langle f|g\rangle' := \frac{1}{n!} \prod_{l=1}^{n} \int_{-1/2}^{1/2} dy_l \prod_{j\ne k}^{n} (\frac{w_j}{w_k};q)_\lambda f^* g.$$

The Fourier formula then gives

$$\sum_{j=1}^{n} \delta(y_j) = \sum_{\kappa} \frac{\beta_\kappa}{\langle P_\kappa | P_\kappa \rangle'} P_\kappa(w_1, \dots, w_n; q, q^\lambda) + \sum_{l=1}^{\infty} \sum_{\kappa:\, \kappa_n = 0} \frac{\gamma_{l,\kappa}}{\langle P_\kappa | P_\kappa \rangle'} \prod_{j=1}^{n} w_j^{-l}\, P_\kappa(w_1, \dots, w_n; q, q^\lambda) \tag{A5}$$

where

$$\beta_\kappa = \frac{n}{n!} \prod_{l=2}^{n} \int_{-1/2}^{1/2} dy_l\, f(1, w_2, \dots, w_n)$$

and similarly for $\gamma_{l,\kappa}$. Thus (A4) immediately gives the value of β_κ.

To derive the formula for the power sum expansion from (A5) we note that

$$\sum_{j=1}^{n} \delta(y_j) = \sum_{k=-\infty}^{\infty} (w_1^k + \cdots + w_n^k).$$

Since $P_\kappa(w_1, \dots, w_n; q, q^\lambda)$ is homogeneous of order $|\kappa|$, it follows by equating terms homogenoeus of order $|\kappa|$ on both sides that

$$w_1^k + \cdots + w_n^k = \sum_{|\kappa| = k} \frac{\beta_\kappa}{\langle P_\kappa | P_\kappa \rangle'} P_\kappa(w_1, \dots, w_n; q, q^\lambda).$$

Substituting n times the r.h.s. of (A4) for β_κ gives the stated result.

We should remark here, that the coefficients relating the power sums $\tilde{p}_k(w_1, \dots, w_n) := \sum_{i=1}^{n} w_i^n$ and the Macdonald polynomials $P_\kappa(w_1, \dots, w_n; q, q^\lambda)$ can also be deduced from certain results in Macdonald's book[12]. Let $t = q^\lambda$ as before, and for a partition $\sigma = (n^{f_n} \dots 2^{f_2} 1^{f_1})$, let

$$z_\sigma(t) = \prod_i i^{f_i} f_i!\, (1 - t^i)^{-f_i}$$

and also define

$$\begin{aligned} c_\kappa(q,t) &= \prod_{s \in \kappa} (1 - q^{a(s)} t^{l(s)+1}) \\ c'_\kappa(q,t) &= \prod_{s \in \kappa} (1 - q^{a(s)+1} t^{l(s)}) \end{aligned} \tag{A6}$$

where $a(s)$ (respectively $l(s)$) are the the number of squares to the right (resp. underneath) the node s in the Ferrer's diagram of κ. Macdonald introduces functions $X_\sigma^\kappa(q,t)$ (which are conjectured to be polynomials in q and t) such that for an arbitrary number of indeterminates w_i,

$$P_\kappa(w; q, t) = \frac{1}{c_\kappa(q,t)} \sum_\sigma \frac{1}{z_\sigma(t)} X_\sigma^\kappa(q,t)\, \tilde{p}_\sigma(w)$$

The polynomials $X_\sigma^\kappa(q,t)$ obey an orthogonality relation which allows us to invert the above equation, yielding

$$\tilde{p}_\sigma(w) = \prod_i (1 - q^i)^{f_i} \sum_\kappa \frac{1}{c'_\kappa(q,t)} X_\sigma^\kappa(q,t)\, P_\kappa(w; q, t)$$

In the particular case $\sigma = (k)$, there is the explicit formula [12, p 366]

$$X_{(k)}^\kappa(q,t) = \prod_{(i,j) \in \kappa} (t^{i-1} - q^{j-1}) \tag{A7}$$

where the product is over all nodes (i,j) in κ (labelled in matrix-fashion) *excluding* the node $(1,1)$. Macdonald has essentially shown that (A6) and (A7) can be re-written in "label-dependent" forms

$$
\begin{aligned}
X^{\kappa}_{(k)}(q,t) &= t^{n(\kappa)}\,(q;q)_{\kappa_1-1}\prod_{i=2}^{r}(t^{-i+1};q)_{\kappa_i} \\
c'_{\kappa}(q,t) &= \prod_{i=1}^{r}\frac{1}{(q;q)_{\kappa_i+\lambda(r-i)}}\prod_{1\le i<j\le r}(q^{\kappa_i-\kappa_j+1+\lambda(j-i-1)};q)_{\lambda}
\end{aligned}
$$

where r is the length of κ (that is, the number of non-zero parts), and $n(\kappa) := \sum_i (i-1)\kappa_i$. It thus follows that $\tilde{p}_k(w) = \sum_{|\kappa|=k} a_\kappa\, P_\kappa(w;q,q^\lambda)$ where

$$
\begin{aligned}
a_\kappa = q^{\lambda\, n(\kappa)}\,[k]_q\,\Gamma_q(\kappa_1)\prod_{i=2}^{r}\frac{\Gamma_q(\lambda(-i+1)+\kappa_i)}{\Gamma_q(\lambda(-i+1))}\prod_{i=1}^{r}\frac{1}{\Gamma_q(\lambda(r-i)+\kappa_i)} \\
\times\prod_{1\le i<j\le r}\frac{\Gamma_q(\kappa_i-\kappa_j+1+\lambda(j-i))}{\Gamma_q(\kappa_i-\kappa_j+1+\lambda(j-i-1))}
\end{aligned}
\tag{A8}
$$

Through simplification one can show that

$$
\frac{\alpha_{\kappa;q}}{\langle P_\kappa|P_\kappa\rangle'} = a_\kappa
$$

thus providing an alternative proof of Proposition A2. Furthermore, the formula (A8) explicitly demonstrates that the coefficients in the expansion are independent of the number of variables n.

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